ON BERNOULLI NUMBERS AND ITS PROPERTIES

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Abstract. In this survey paper, I first review the history of Bernoulli numbers, then examine the modern definition of Bernoulli numbers and the appearance of Bernoulli numbers in expansion of functions. I revisit some properties of Bernoulli numbers and the history of the computation of big Bernoulli numbers.

1. Introduction

Two thousand years ago, Greek mathematician Pythagoras first noted about triangle numbers which is $1+2+3+\cdots+n$. Archimedes found out

$$1^2+2^2+3^2+\cdots+n^2=\frac{1}{6}n(n+1)(2n+1). \tag{1}$$

Later in the fifth century, Indian mathematician Aryabhata proposed

$$1^3+2^3+3^3+\cdots+n^3=[\frac{1}{2}n(n+1)]^2 \tag{2}$$

which Jacobi gave the first vigorous proof in 1834. It is not until five hundred years later that Arabian mathematician Al-Khwarizm showed

$$1^4+2^4+3^4+\cdots+n^4=\frac{1}{30}n(n+1)(2n+1)(3n^2+3n-1). \tag{3}$$

Studies the more generalized formula for $\sum_{k=1}^{n-1}k^r$ for any natural number r was only carried out in the last few centuries. Among them,the investigation of Bernoulli numbers is much significant.

In this paper, I present an elementary examination of the development of Bernoulli numbers and a concise review of its appearance in expansions of various functions . I also aim to explore its properties and its applications in other fields of mathematics [1].

2. Bernoulli Numbers

Swiss mathematician Jakob Bernoulli (1654-1705)once claimed that instead of laboring for hours to get a sum of powers, he only used several minutes to calculate sum of powers such as $1^{10}+2^{10}+3^{10}+\cdots+1000^{10}=91,409,924,241,424,243,424,241,924,242,500$. Obviously, he had used a summation formula, knowing the first 10 Bernoulli numbers [2].

It was already known to Jakob Bernoulli that

$$\sum_{k=1}^{n-1} k = \frac{1}{2}n(n-1) = \frac{1}{2}n^2 - \frac{1}{2}n \tag{4}$$

$$\sum_{k=1}^{n-1} k^2 = \frac{1}{6}n(n-1)(2n-1) = \frac{1}{3}n^3 - \frac{1}{2}n^2 + \frac{1}{6}n \tag{5}$$

$$\sum_{k=1}^{n-1} k^4 = \frac{1}{4}n^2(n-1)^2 = \frac{1}{4}n^4 - \frac{1}{2}n^3 + \frac{1}{4}n^2 \tag{6}$$

$$\begin{aligned}\sum_{k=1}^{n-1} k^4 = \frac{1}{30}n(n-1)(2n-1)(3n^2-3n-1) &= \frac{1}{5}n^5 - \frac{1}{2}n^4 + \\ &+ \frac{1}{3}n^3 - \frac{1}{30}n\end{aligned} \tag{7}$$

$$\begin{aligned}\sum_{k=1}^{n-1} k^5 = \frac{1}{12}n^2(2n^2-2n-1)(n-1)^2 &= \frac{1}{6}n^6 - \frac{1}{2}n^5 + \\ &+ \frac{5}{12}n^4 - \frac{1}{12}n^2\end{aligned} \tag{8}$$

More generally, the $\sum_{k=1}^{n-1} k^r$ can be written in the form of

$$\begin{aligned}\sum_{k=1}^{n-1} k^r &= \sum_{k=0}^{r} \frac{B_k}{k!}\frac{r!}{(r-k+1)!}n^{r-k+1} \\ &= \frac{B_0}{0!}\frac{n^{r+1}}{r+1} + \frac{B_1}{1!}n^r + \frac{B_2}{2!}rn^{r-1} + \cdots + \frac{B_r}{r!}n\end{aligned} \tag{9}$$

where the B_k numbers which are independent of r and called Bernoulli's numbers. The first few Bernoulli numbers B_n are

$$\begin{aligned}B_0 &= 1 \\ B_1 &= -\frac{1}{2} \\ B_2 &= \frac{1}{6} \\ B_4 &= -\frac{1}{30} \\ B_6 &= \frac{1}{42} \\ B_8 &= -\frac{1}{30}\end{aligned}$$

$$\begin{aligned}
B_{10} &= \frac{5}{66}\\
B_{12} &= -\frac{691}{2,730}\\
B_{14} &= \frac{7}{6}\\
B_{16} &= -\frac{3,617}{510}\\
B_{18} &= \frac{43,867}{798}\\
B_{20} &= -\frac{174,611}{330}\\
B_{22} &= \frac{854,513}{138}\\
B_{24} &= -\frac{236,364,091}{2,730}\\
B_{26} &= \frac{8,553,103}{6}\\
B_{28} &= -\frac{23,749,461,029}{870}\\
B_{30} &= \frac{8,615,841,276,005}{14,322}\\
B_{32} &= -\frac{7,709,321,041,217}{510}\\
B_{34} &= \frac{2,577,687,858,367}{6}\\
B_{36} &= -\frac{26,315,271,553,053,477,373}{1,919,190}\\
B_{38} &= \frac{2,929,993,913,841,559}{6}\\
B_{40} &= -\frac{261,082,718,496,449,122,051}{13,530}
\end{aligned}$$

with

$$B_{2n+1} = 0 \tag{10}$$

for all positive integer n. More numbers are given in [3].

3. **Expansion of Usual Functions**

An equivalent definition of the Bernoulli's numbers is obtained from the series expansion of the identity

$$\frac{x}{e^x-1} \equiv \sum_{n=0}^{\infty} \frac{B_n x^n}{n!}. \tag{11}$$

This leads to

$$\frac{x}{e^x-1}+\frac{x}{2} = \frac{x}{2}\left(\frac{2}{e^x-1}+1\right)=\frac{x}{2}\frac{e^x+1}{e^x-1}=\frac{x}{2}coth\left(\frac{x}{2}\right). \tag{12}$$

It can be rewritten as

$$\frac{x}{2}coth(\frac{x}{2}) = \sum_{n=0}^{\infty} \frac{B_{2n}x^{2n}}{(2n)!}. \tag{13}$$

If substitute x with $2ix$, then it gives

$$xcot(x) = \sum_{n=0}^{\infty}(-1)^n \frac{B_{2n}(2x)^{2n}}{(2n)!} \tag{14}$$

for $x \in [-\pi, \pi]$. Thus the following expansions are obtained,

$$coth(x) = \sum_{n=0}^{\infty} \frac{2B_{2n}(2x)^{2n-1}}{(2n)!} \tag{15}$$

$$cot(x) = \sum_{n=0}^{\infty}(-1)^n \frac{2B_{2n}(2x)^{2n-1}}{(2n)!} \tag{16}$$

Now it's also possible to find the expansion for $tanh(x)$ and $tan(x)$. As I observe that

$$\begin{aligned} 2coth(2x) - coth(x) &= 2\frac{cosh(2x)}{sinh(2x)} - \frac{cosh(z)}{sinh(z)} \\ &= \frac{cosh^2(x) + sinh^2(x)}{sinh(x)cosh(x)} - \frac{cosh(x)}{sinh(x)} = tanh(x), \end{aligned}$$

I can have

$$tanh(x) = \sum_{n=1}^{\infty} \frac{2(4^n-1)B_{2n}(2x)^{2n-1}}{(2n)!}, \tag{17}$$

and then,

$$tan(x) = \sum_{n=1}^{\infty}(-1)^n \frac{2(1-4^n)B_{2n}(2x)^{2n-1}}{(2n)!} \tag{18}$$

both for $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$ Bernoulli's numbers also occur in the expansions of other classical functions such as $csc(x)$,$csch(x)$,$ln|sin(x)|$,$ln|cos(x)|$,$ln|tan(x)|$, $\frac{x}{sinh(x)}$, and etc.

Another intriguing fact is that zeta function $\zeta(2k)$ for any natural number k can also be expressed in Bernoulli numbers [4]

$$\zeta(2k) = \sum_{n=1}^{\infty} \frac{1}{n^{2k}} = \frac{4^k|B_{2k}|\pi^{2k}}{2(2k)!} \tag{19}$$

However, there is no similar expression known for $\zeta(2k+1)$. Rearrange the equation, I get

$$B_{2k} = \frac{(-1)^{k-1}2(2k)!}{(2\pi)^{2k}} \sum_{n=1}^{\infty} \frac{1}{n^{2k}} = \frac{(-1)^{k-1}2(2k)!}{(2\pi^{2k})}\zeta(2k). \tag{20}$$

This expression gives rise to an approximation of Bernoulli numbers. When k becomes large,

$$\zeta(2k) \simeq 1, \tag{21}$$

while with Stirling's formula

$$(2k)! \simeq (2k)^{2k} e^{-2k} \sqrt{4\pi k}, \tag{22}$$

so I have

$$B_{2k} \simeq (-1)^{k-1} 4 (\frac{k}{\pi e})^{2k} \sqrt{\pi k}. \tag{23}$$

4. Properties of Bernoulli Numbers

The Bernoulli numbers are given by the double sum

$$B_n = \sum_{k=0}^{n} \frac{1}{k+1} \sum_{r=0}^{k} (-1)^r \binom{k}{r} r^n \tag{24}$$

where $\binom{k}{r}$ is a binomial coefficient. They also satisfy the following interesting summations [5, 6]

$$\sum_{k=0}^{n-1} \binom{n}{k} B_k = 0, \tag{25}$$

$$\sum_{k=0}^{n} \binom{6n+3}{6k} B_{6k} = 2n+1, \tag{26}$$

$$\sum_{k=0}^{n} \binom{6n+5}{6k+2} B_{6n+2} = \frac{1}{3}(6n+5). \tag{27}$$

The famous Clausen-von Staudt's theorem regarding Bernoulli numbers' fractional part was published by Karl von Staudt (1798-1867) and Thomas Clausen (1801-1885) independently in 1840. It allows to compute easily the fractional part of Bernoulli's numbers and thus also permits to compute the denominator of those numbers. It says, the value B_{2k}, added to the sum of the inverse of prime numbers p such that $(p-1)$ divides $2k$, is an integer [7]. In other words,

$$-B_{2k} \equiv \sum_{(p-1)|2k} \frac{1}{p} (mod 1) \tag{28}$$

For example, $k = 8$,

$$\begin{aligned} -B_{16} &\equiv \sum_{(p-1)|16} \frac{1}{p} \equiv \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{17} \equiv \frac{47}{510} (mod 1) \\ &\Rightarrow \quad B_{16} \equiv \frac{463}{510} (mod 1). \end{aligned}$$

One of the easy consequences of the Staudt's theorem is that for every prime numbers k of the form $3n+1$

$$B_{2k} \equiv \frac{1}{6} (mod 1). \tag{29}$$

This is because that $p-1$ divides $2k = 2(3n+1)$ only if $p-1$ is one of 1, 2, $3n+1$, $6n+2$, that is p is one of 2, 3, $3n+2$, $6n+3$. But $6n+3$ is divisible by 3 and $3n+2$ is divisible by 2 because 3n+1 is prime so the only primes

p candidates are 2 and 3. The first primes of the form 3n+1 are 7, 13, 19, 31, 37, 43, 61, 67, 73, 79, 97, 103... hence I have

$$\begin{aligned} B_{14} &\equiv B_{26} \equiv B_{38} \equiv B_{62} \equiv B_{74} \equiv B_{86} \equiv B_{122} \\ &\equiv B_{134} \equiv B_{146} \equiv B_{158} \equiv B_{194} \equiv B_{206} \equiv \cdots \equiv \frac{1}{6}(mod 1) \end{aligned}$$

Staudt's theorem is very useful and significant in the sense that it permits to compute exactly a Bernoulli's number as soon as there is a sufficiently good approximation of it.

Besides its applications in series expansions, Bernoulli numbers are also widely used in differential topology, mathematical analysis, and number theory. Interestingly, it is also related to the famous Fermat's last theorem.

Fermat's Last Theorem states

$$x^n + y^n = z^n$$

has no non-zero integer solutions for $n > 2$. Ever since Fermat expressed this result around 1630, generations of mathematicians have dived enthusiastically into the pursuit of a vigorous proof .

A breakthrough was made in 1850 by Ernst Kummer (1810-1893) when he proved Fermat's theorem for n=p, whenever p is a regular prime. Kummer gave the beautiful regularity criterion:

p is a regular prime if and only if p does not divide the numerator of $B_2, B_4, ..., B_{p-3}$.

He showed that all primes before 37 where regular, hence Fermat's theorem was proved for those primes. 37 is the first non regular prime because it divides the numerator of

$$B_{32} = \frac{7709321041217}{510} = \frac{37208360028141}{510}$$

The next irregular primes (less than 300) are

$$59, 67, 101, 103, 131, 149, 157, 233, 257, 263, 271, 283, 293, ...$$

For example, 157 divides the numerators of B_{62} and B_{110}.

By applying arithmetical properties of Bernoulli's numbers, Johann Ludwig Jensen (1859-1925) proved in 1915 that the number of irregular primes is infinite [8].

5. Computation of Bernoulli Numbers

Bernoulli himself calculated the numbers up to B_{10}. Later, Euler worked up to B_{30}. One century later, Adams made the computa of all Bernoulli's numbers up to B_{124} [9]. In 1996, Simon Plouffe and Greg J. Fee computed $B_{200,000}$, and this huge number has about 800,000 digits. In July 10th 2002, they improved the record to $B_{750,000}$ which has 3,391,993 digits by a 21 hours computation on their personal computer [10]. The method is based

on the relation between zeta function and Bernoulli numbers, which allow a direct computation of the target number without the need of calculating the previous numbers.

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